DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 98 (2000) 173–190

An alternate formulation of the symmetric traveling salesman problem and its properties

T.S. Arthanari [1], M. Usha *

Indian Statistical Institute, 110, Nelson Manickam Road, Chennai 600029, India

Received 10 November 1997; revised 22 December 1997; accepted 12 April 1999

Abstract

In this paper we give an alternate formulation of the symmetric traveling salesman problem and give its properties. We compare the polytope defined by this formulation, $\mathscr{U}(n)$, with the standard subtour elimination polytope $SEP(n)$. We show $\mathscr{U}(n) \subseteq SEP(n)$.

Keywords: Traveling salesman problem; Subtour elimination polytope; Problem formulations

1. Introduction

The traveling salesman problem(TSP) is one of the most extensively studied combinatorial optimisation problems. The symmetric traveling salesman polytope, $\mathscr{Q}(n)$, is defined as the convex hull of the incidence vectors of edge sets of Hamiltonian cycles on a complete graph on n vertices. Over the years researchers have given different formulations to solve the TSP as a linear programming problem. Finding complete descriptions of the polytope $\mathscr{Q}(n)$ has also proved to be a challenging problem and to date complete descriptions of $\mathscr{Q}(n)$ is available for $n \leqslant 8$. We refer to [8,7] for a detailed survey of this problem.

Dantzig et al. [4] formulated the asymmetric traveling salesman problem as a 0–1 linear program on a graph (V,E). Their formulation for the symmetric case gives rise to the standard subtour elimination polytope $SEP(n)$. Many formulations have proposed since, some in higher dimensions with more variables and fewer constraints, for the same problem. Padberg and Sung [9] give an analytical comparison of different formulations and have remarked that the DFJ formulation is the best so far. They proposed

* Corresponding author. Address: Plot no. 406, Ram Nagar, Velachery, Chennai, 600 042, India.
E-mail addresses: t.arthanari@auckland.ac.nz (T.S. Arthanari), umohan@hotmail.com (M. Usha)
[1] Currently at Department of Engineering Science, University of Auckland, Auckland, New Zealand.

0166-218X/00/$ - see front matter
PII: S0166-218X(99)00154-7

comparing the polytopes defined by the formulations as a measure of comparison. If $\mathscr{X}_{\mathrm{A}}$ is the polytope defined by formulation A and $\mathscr{X}_{\mathrm{B}}$ is the polytope defined by formulation B, both in the same space of variables, formulation A is '*better than*' formulation B if $\mathscr{X}_{\mathrm{A}} \subset \mathscr{X}_{\mathrm{B}}$. In case we have another formulation C for the same problem but in a different dimension defining polytope $\mathscr{Z}_{\mathrm{C}}$. Then formulation A is '*better than*' formulation C if $\mathscr{X}_{\mathrm{A}} \subset T(\mathscr{Z}_{\mathrm{C}})$, where T is an affine transformation mapping integer(mixed integer) points of $\mathscr{Z}_{\mathrm{C}}$ onto the space where $\mathscr{X}_{\mathrm{A}}$ is defined. This affine transformation is bijective on the integer points.

Gouveia and Voß [5] compare six formulations of the time-dependent TSP (TDTSP) problem and show which of them are stronger formulations. They use the approach of showing certain constraints of one formulation imply one or more constraints of another formulation. Also they exhibit a point which is in one of the feasible regions but not in the other, to imply strict inclusion. However none of the formulations they consider imply the subtour elimination constraints of the DFJ type.

Arthanari [1] posed the symmetric traveling salesman problem(STSP) as a multistage decision problem and gave a mathematical programming formulation of the same. He showed that the slack variables that arise of this formulation are precisely the edge-tour incidence vectors. This formulation uses $\mathcal{O}(n^3)$ variables but only a quadratic number of constraints. Bellman [3] and Held and Karp [6] were the first to consider multi-stage decision (dynamic programming) approach to TSP. However their formulation is different from that of Arthanari [1].

In this paper we show that the formulation we have, defines the $SEP(n)$ $\forall n$ using polynomial number of constraints. In Section 2 we give preliminaries of the STSP and the subtour elimination polytope ($SEP(n)$). In Section 3 we give notations and definitions used. In Section 4 we give the mathematical programming formulation of the STSP as a multistage decision problem. We state some properties and give results on this formulation. We show $\mathscr{U}(n) \subseteq SEP(n)$ $\forall n$. In Section 5 we draw conclusions from our work.

2. Preliminaries

Let $G=(V,E)$ be a graph. An edge $e \in E$ is an unordered pair $[u,v]$ of nodes of V, $E \subseteq V^2$, where V^2 is the set of all unordered pairs of distinct elements of V. A weighted graph is a pair (G,x) where $G=(V,E)$ is a graph and x is a vector of R^{V^2}. Given a subset of nodes $W \subseteq V$, we define the following sets:

$$E(W)=\{[u,v] \in E\colon u \in W, v \in W\},$$
$$\delta(W)=\{[u,v] \in E\colon u \in W, v \notin W\}.$$

For singleton sets $\delta(\{w\})=\delta(w)$. Given a vector $x \in R^E$, x_e *or* $x_{[u,v]}$ denotes the component of x corresponding to the edge $e \in E$. For any subset $F \subseteq E$ we denote the indicator of F by x^F, where

$$x_e^F = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Also we define $x(F) = \sum_{e \in F} x_e$ for any vector $x \in R^E$.

Given two nonempty sets $S \subseteq V$ and $T \subseteq V - S$ we define a cut as

$$\{S : T\} = \{[u,v] \in E\colon u \in S, v \in T\}.$$

Let $K_n = (V_n, E_n)$ be a complete graph on n vertices. Let c_{ij} be the weight on edge $[i,j]$.

Definition 2.1. A Hamiltonian cycle or tour of a graph is a cycle that contains all nodes of the graph.

Thus given a graph $G = (V,E)$ the edge set $E' \subseteq E$ represents a tour $\Leftrightarrow$ the subgraph $G' = (V,E')$ is connected and each node is met by exactly two edges in G'.

The symmetric traveling salesman problem consists of finding a Hamiltonian cycle, H, of K_n which minimises $c(H)$, where c is the given weight on the edges. Let $\mathscr{T}_n$ be the set of all tours in K_n. Then the polytope

$$\mathscr{Q}(n) = conv\{x^T \in R^{E_n}\colon T \in \mathscr{T}_n\}$$

is called the symmetric traveling salesman polytope and dim $(\mathscr{Q}(n)) = (n(n-3))/2$. For $n = 3$, $\mathscr{T}_n$ is a singleton. Henceforth we assume $n \geqslant 4$.

SUBTOUR ELIMINATION POLYTOPE *SEP*(n):

SEP(n) is the polytope defined by the set of all $x \in R^{E_n}$ such that (2.1), (2.2) and (2.3) hold:

$$x_e \geqslant 0, \quad \forall e \in E_n, \tag{2.1}$$

$$x(\delta(v)) = 2, \quad \forall v \in V_n, \tag{2.2}$$

$$x(\delta(S)) \geqslant 2, \quad \forall S \subseteq V_n, \quad S \neq \phi \textit{ and } S \neq V_n. \tag{2.3}$$

The dimension of this polytope is $(n(n-1))/2$ and the number of constraints is exponential in this case. Also $\mathscr{Q}(n) \subseteq SEP(n)\ \forall n$.

3. Notations and definitions

Let n denote the number of cities.

Definition 3.1. $t = (1, i_1, \ldots, i_{k-1}, 1)$ is a k-tour in case $(i_1, \ldots, i_{k-1})$ is a permutation of $(2, \ldots, k)$, $k \leqslant n$.

Let $\mathscr{C}_{ijk} = c_{ik} + c_{jk} - c_{ij}$ *for* $4 \leqslant k \leqslant n$, $1 \leqslant i < j \leqslant k-1$.

Definition 3.2. The length of a k-tour is defined as $\mathscr{C}(t)$ given by

$$\mathscr{C}(t)=\sum_{r=1}^{k-2} c_{i_r i_{r+1}} + c_{1.i_1} + c_{i_{k-1}1}.$$

Let $\mathscr{T}_k$ denote the set of all k-tours and $\mathscr{T}_{ijk}$ denote the set of all k-tours in which edge$[i,j]$ appears that is i and j are adjacent to each other in every k-tour in $\mathscr{T}_{ijk}$. Then we have $\mathscr{T}_k=\bigcup_{1\leqslant i<j\leqslant k}\mathscr{T}_{ijk}$.

Let F_{ij}^k be a mapping from $\mathscr{T}_{ijk-1}$ to $\mathscr{T}_k$ such that for $t\in\mathscr{T}_{ijk}$, $t=(1,i_1,\ldots,i,j,\ldots,i_{k-1},1)$; $F_{ij}^k(t)=(1,i_1,\ldots,i,k,j,\ldots,i_{k-1},1)\in\mathscr{T}_k$, i.e., $F_{ij}^k(t)$ is the k-tour obtained from the $(k-1)$-tour t by inserting k between i and j.

We start with the 3-tour $t=(1,2,3,1)$.

Example 3.1. Take $n=5$. Consider $t=(1,4,3,2,1)\in\mathscr{T}_4$. Then t belongs to each one of $\mathscr{T}_{144},\mathscr{T}_{344},\mathscr{T}_{234},\mathscr{T}_{124}.F_{14}^5(T)=(1,5,4,3,2,1)\in\mathscr{T}_5$.

We now state some results.

Proposition 3.1. Let $t_1,t_2\in\mathscr{T}_{ijk-1}$. If $\mathscr{C}(t_1)\leqslant\mathscr{C}(t_2)$ then

$$\mathscr{C}(F_{ij}^k(t_1))\leqslant\mathscr{C}(F_{ij}^k(t_2)).$$

Proposition 3.2. $\mathscr{T}_{k+1}=\bigcup_{1\leqslant i<j\leqslant k}\{F_{ij}^{k+1}(t)\backslash t\in\mathscr{T}_{ijk}\}$.

Proposition 3.3. $\min_{t\in\mathscr{T}_{k+1}}\mathscr{C}(t)=\min_{1\leqslant i<j\leqslant k}\{\min_{t\in\mathscr{T}_{ijk}}\mathscr{C}(t)+\mathscr{C}_{ijk+1}\}$.

Remark 3.1. The symmetric traveling salesman problem is to find an optimal n-tour, given c_{ij}, $1\leqslant i<j\leqslant n$, with $c_{ij}=c_{ji}$. Proposition 3.3 assures an optimal n-tour, if we have a subset of $(n-1)$ tours which includes for each $1\leqslant i<j\leqslant n-1$, $a(n-1)$ tour in which i and j are adjacent and it minimises the length of the tour among all such $(n-1)$ tours in which i and j are adjacent. However, finding such $(n-1)$ tours may not be an easy task.

Thus we really have a $(n-3)$ stage decision problem, in which in stage $(k-3)$, $4\leqslant k\leqslant n$, we decide on where to insert k. In the beginning we have a 3-tour $(1,2,3,1)$. In the first stage we decide on where to insert 4 among the available pairs $[1,2]$, $[2,3]$, *and* $[1,3]$. Depending on this decision we have certain available pairs for the second stage insertion.

In the second stage we decide on where to insert 5 among the available pairs. For instance, our decision in the first stage is to introduce 4 between i_4 *and* j_4. Then the available pairs are

$$A_5=\{[1,2],[1,3],[2,3]\}\cup\{[i_4,4],[j_4,4]\}-\{[i_4,j_4]\}.$$

In general A_k depends on the decisions made in preceding stages, $4\leqslant k\leqslant n$. We have

$$A_k=A_{k-1}\cup\{[i_{k-1},k-1],[k-1,j_{k-1}]\}-\{[i_{k-1},j_{k-1}]\}$$

for some $[i_{k-1}, j_{k-1}] \in A_{k-1}$. A_k gives the set of all $[i,j]$ such that they are adjacent in the $(k-1)$ tour, which results out of the decisions made in the preceding stages.

The associated total cost of these decision made at different stages is

$$\mathscr{C}_{i_4 j_4 4} + \mathscr{C}_{i_5 j_5 5} + \cdots + \mathscr{C}_{i_n j_n n}.$$

We are interested in finding optimal $[i_4, j_4], \ldots, [i_n, j_n]$ such that the total cost is minimum. This finally produces an n-tour. The length of this tour is given by $(c_{12} + c_{13} + c_{23}) + \sum_{k=4}^{n} \mathscr{C}_{i_k j_k k}$. Here $(c_{12} + c_{13} + c_{23})$ is the length of the initial 3-tour which is independent of the decisions subsequently made.

4. Mathematical programming formulation of the STSP

In this section we describe the 0–1 integer programming formulation of the STSP given by Arthanari [1].

Let

$$x_{ijk} = \begin{cases} 1 & \text{if in stage } (k-3) \text{ the decision is to insert } k \text{ between } i \text{ and } j, \\ & \quad 1 \leqslant i < j \leqslant k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.1. Given $X = (x_{124}, \ldots, x_{n-2,n-1,n}) \in B^{\tau} = \{0,1\}^{\tau}$ where $\tau = \sum_{4}^{n} \frac{(k-1)(k-2)}{2}$. We say X is a feasible decision vector in case,
(i) For every $k = 4, \ldots, n$

$$\sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} = 1 \tag{4.1}$$

that is k is inserted between i and j for exactly one pair $[i,j]$. and
(ii) $x_{ijk} = 1 \Rightarrow T_{k-1}(X) \in \mathscr{T}_{ijk-1}$, where $T_{k-1}(X)$ is the $(k-1)$-tour resulting from the preceding decisions, that is, depending on $(x_{124}, \ldots, x_{k-3,k-2,k-1})$, denoted by $X/k-1$.

In other words, X is a feasible decision vector if $x_{i_k j_k k} = 1 \Rightarrow [i_k, j_k] \in A_k,\ 4 \leqslant k \leqslant n$.

Example 4.1. For $n = 6$, let $x_{124} = 1$, $x_{145} = 1$ & $x_{236} = 1$, then $X = (100;\ 000100;\ 0010000000)$ is a feasible decision vector as

$$\sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} = 1 \quad \text{for } k = 4, 5, \text{ and } 6 \tag{4.2}$$

and $x_{124} = 1$, requires $T_3(X) \in \mathscr{T}_{123}$. And this is true as

$$T_3(X) = (1, 2, 3, 1).$$

Similarly $x_{145} = 1 \Rightarrow T_4(X) = (1,4,2,3,1) \in \mathscr{T}_{144}$ and $x_{236} = 1 \Rightarrow T_5(X) = (1,5,4,2,3,1) \in \mathscr{T}_{235}$.

However, $X = (100, 100000, 0010000000)$ is not a feasible decision vector as $T_4(X) = (1,4,2,3,1)$, resulting from $x_{124} = 1 \notin \mathscr{T}_{124}$ as required for $x_{125} = 1$.

Let $\mathfrak{I}$ be the set of all feasible decision vectors. We can state the multistage decision process as

Problem 0. Find $X^* \in \mathfrak{I}$ such that $\mathscr{C}(X^*) = \min_{X \in \mathfrak{I}} \mathscr{C}(X)$ where

$$\mathscr{C}(X) = \sum_{k=4}^{n} \sum_{1 \leqslant i < j \leqslant k-1} \mathscr{C}_{ijk} x_{ijk}.$$

We shall now show, how $X \in$ can be expressed as a set of linear equalities and inequalities along with $X \in B^\tau$.

Notice that we already have

$$\sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} = 1 \quad \text{for } X \in \mathfrak{I}. \tag{4.3}$$

In addition, x_{ijk} cannot be 1 if $[i,j] \notin T_{k-1}(X)$.

Condition (ii) of Definition 4.1 states that $x_{i_k j_k k} = 1 \Rightarrow [i_k, j_k] \in A_k$; $4 \leqslant k \leqslant n$. We express this as linear inequality as follows:

For all X, we have $[1,2], [1,3] \& [2,3] \in T_3(X)$ as the initial tour is always $(1,2,3,1)$. And the edges are available in all sets A_k, $4 \leqslant k \leqslant n$ unless $x_{ijk} = 1$. Since we begin with the 3-tour and at most one of the $x_{ijk} = 1$, $4 \leqslant k \leqslant n$ for each $[i,j]$; $1 \leqslant i < j \leqslant 3$ we have the following constraint

$$\sum_{4 \leqslant k \leqslant n} x_{ijk} \leqslant 1. \tag{4.4}$$

Now consider other $[i,j]'s$, *for* $4 \leqslant j \leqslant n-1$ and $1 \leqslant i < j$. x_{ijk} cannot be 1 unless $[i,j]$ is an edge in the $(k-1)$-*tour* resulting from earlier decisions given by $X/k-1$. However $[i,j]$ is created only in one of the two ways, given below:

Either (i) $x_{rij} = 1$ for some $1 \leqslant r < i$ or (ii) $x_{isj} = 1$ for some $i+1 \leqslant s < j$. Therefore, if

$$\sum_{1 \leqslant r \leqslant i-1} x_{rij} + \sum_{i+1 \leqslant s \leqslant j-1} x_{isj} = 1, \tag{4.5}$$

then edge $[i,j]$ is present at the kth stage and hence x_{ijk} can either be 0 or 1 for any $k \geqslant j+1$. If

$$\sum_{1 \leqslant r \leqslant i-1} x_{rij} + \sum_{i+1 \leqslant s \leqslant j-1} x_{isj} = 0, \tag{4.6}$$

then the edge $[i,j]$ is not available for insertion from the k^{th} stage; $k \geqslant j+1$ and $\sum_{j+1 \leqslant k \leqslant n} x_{ijk} = 0$. Hence we have

$$\sum_{j+1 \leqslant k \leqslant n} x_{ijk} \leqslant \sum_{1 \leqslant r \leqslant i-1} x_{rij} + \sum_{i+1 \leqslant s \leqslant j-1} x_{isj}$$

$$\Rightarrow -\sum_{1 \leqslant r \leqslant i-1} x_{rij} - \sum_{i+1 \leqslant s \leqslant j-1} x_{isj} + \sum_{j+1 \leqslant k \leqslant n} x_{ijk} \leqslant 0. \tag{4.7}$$

Now Problem 0 can be given a 0–1 programming formulation as given below:

Problem 1.

$$\begin{aligned}
&\text{minimise} \quad \sum_{k=4}^{n} \sum_{1\leqslant i<j\leqslant k-1} \mathscr{C}_{ijk} x_{ijk} \\
&\text{subject to} \quad \sum_{1\leqslant i<j\leqslant k-1} x_{ijk} = 1, \quad 4\leqslant k\leqslant n,
\end{aligned} \tag{4.8}$$

$$\sum_{k=4}^{n} x_{ijk} \leqslant 1, \quad 1\leqslant i < j\leqslant 3, \tag{4.9}$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} \leqslant 0, \quad 4\leqslant j\leqslant n-1, \quad 1\leqslant i<j, \tag{4.10}$$

$$x_{ijk} = 0 \text{ or } 1, \quad 1\leqslant i < j\leqslant k, \quad 4\leqslant k\leqslant n. \tag{4.11}$$

Remark 4.1. The objective function is same as in Problem 0.

Let $\boldsymbol{E_n}$ denote the matrix corresponding to the constraints given by (4.8). $\boldsymbol{E_n}$ is a $(n-3)\times\tau$ matrix of the following form:

$$\boldsymbol{E_n} = \begin{bmatrix} e_{\frac{3\times 2}{2}} & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & e_{\frac{(n-1)(n-2)}{2}} \end{bmatrix},$$

where e_k is a vector each of whose coordinates is 1.

Let $\bar{\boldsymbol{A}}$ be the matrix of coefficients corresponding to constraint set (4.8)–(4.10). Relaxing the integer constraints and adding the following constraints:

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-1} x_{isn} \leqslant 0, \quad i = 1,\dots,n-1, \tag{4.12}$$

we get the following problem.

Problem 2.

$$\begin{aligned}
&\min \quad \mathscr{C}'X \\
&\text{s.t.} \quad \begin{bmatrix} \boldsymbol{E_n} & \boldsymbol{0} \\ \boldsymbol{A} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} e_{n-3} \\ e_3 \\ 0 \end{bmatrix}, \\
&\quad X, U \geqslant 0.
\end{aligned} \tag{4.13}$$

Note that (4.12) are always satisfied as x_{ijk} are non-negative. However adding these constraints help us bring out the connection between the slack variables of Problem 2 and the edge-tour incident vectors of n-tours given by integer X feasible to Problem 2. Here $\boldsymbol{A}$ is the matrix corresponding to constraints (4.8)–(4.12) without constraint (4.11) and $\boldsymbol{E_n}$ is defined as before.

Theorem 4.1. *Any integer feasible solution to* Problem 2 *is a basic solution and has the following property.*

Let the submatrix of $\boldsymbol{A}$ *corresponding to the columns of* $x_{i_k j_k k} = 1$, $k = 4,\dots,n$ *be denoted by* $\boldsymbol{Q}$. *Then any row of* $\boldsymbol{Q}$ *is such that either*

(i) *All columns in a row are zeroes.*

or (ii) *Exactly one of the elements is* $+1$ *and the rest are zeroes in the row.*

or (iii) *There is a* -1 *and a* $+1$ *in the row and the rest are zeroes.*

or (iv) *There is a* -1 *in the row and the rest are zeroes.*

Moreover any such solution corresponds to a n-tour.

Proof. Consider the square matrix $\boldsymbol{B}$ obtained by taking the columns corresponding to $x_{i_k j_k k} = 1; 4 \leqslant k \leqslant n$ and the columns corresponding to the slack variables u_{ij}. We have

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{Q} & \boldsymbol{I} \end{bmatrix},$$

where $\boldsymbol{Q}$ is the submatrix of $\boldsymbol{A}$ corresponding to the columns $x_{i_k j_k k} = 1; 4 \leqslant k \leqslant n$.

$$\boldsymbol{B}^{-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{Q} & \boldsymbol{I} \end{bmatrix}.$$

Let Q_{ij} denote the row corresponding to the pair $[i, j]$.

Case (i): $1 \leqslant i < j \leqslant 3$.

In this case either no x_{ijk} is positive for pair $[i, j]$ or at most one of them is equal to 1 in any integer feasible solution. This implies either

(a) Q_{ij} is a zero vector where we have an instance of (i) or (b) Q_{ij} has a single 1 and rest zeroes, where we have an instance of (ii). in fact in these rows there can be no -1's.

Case (ii): $1 \leqslant i < j; 4 \leqslant j \leqslant n-1$.

Using the fact that for any $[i, j]$ at most one of the x_{ijk} can be equal to 1 in any integer feasible solution to the problem, there can be at most one $+1$ in any of these rows.

However, this $+1$ cannot occur without a -1 in the same row since

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} \leqslant 0. \tag{4.14}$$

If all x_{rij} or $x_{isj} = 0$ then $\sum_{k=j+1}^{n} x_{ijk} = 1$ and cannot satisfy this constraint. So at least one of the x_{rij} or $x_{isj} = 1$. But for any k at most one $x_{i_k j_k k} = 1$. So there is exactly one -1 in row Q_{ij}. This leads to an instance of (iii). On the other hand, if x_{rij} as well as x_{isj} are zeroes then x_{ijk} must all be zeroes. We have an instance of (i).

Finally if one of the x_{rij} or $x_{isj} = 1$ and all $x_{ijs} = 0$ we have an instance of (iv).

Now we prove that any such solution corresponds to a tour. Consider $x_{i_k j_k k} = 1; 4 \leqslant k \leqslant n$. Insert in the 3-tour $(1,2,3,1)$, city 4 between $[i_4, j_4]$ and obtain a 4-tour. Assume introducing $5,\ldots,k$ in this manner in the $4,\ldots,k-1$ tour respectively we obtain a k tour. We shall show that introducing $(k+1)$ in the unique k tour obtained will result in a $(k+1)$-*tour*.

We need to show that $[i_{k+1}, j_{k+1}]$ is a pair available in

$$A_k(X) \stackrel{\text{def}}{=} \{[1,2],[1,3],[2,3]\} \bigcup_{r=4}^{k} \left\{ [i_r, r], [j_r, r]\} - \bigcup_{r=4}^{k} \{[i_r, j_r]\right\},$$

$$1 \leqslant i_{k+1} < j_{k+1} \leqslant k.$$

If $[i_{k+1}, j_{k+1}] \notin A_k(X)$, Then it must be either
(a) be $[i_r, j_r]$ for some $4 \leqslant r \leqslant k$ or
(b) $[i_{k+1}, j_{k+1}]$ is $[i, r]$ with $i_r \neq i \neq j_r, 4 \leqslant r \leqslant k$.

However, (a) cannot happen as for any pair $[i,j]$, $x_{ijk} = 1$ for at most one r and already $x_{i_r, j_r, r} = 1, 4 \leqslant r \leqslant k$.

If (b) happens then the constraint corresponding to $[i, r]$ will be violated and X cannot be feasible for the problem. This leads to a contradiction. Hence $[i_{k+1}, j_{k+1}] \in A_k(X)$. Hence any such solution corresponds to a tour. Hence the result.

Note 4.1. Any n-tour corresponds to an integer solution to Problem 2. Thus there is a 1–1 correspondence between n-tours and the integer feasible solutions to Problem 2.

Lemma 4.1. *Let U denote the vector of slack variables in Problem 2. Let (X, U) be any integer feasible solution to Problem 2. Then U is the edge-tour incidence vector of the n-tour given by (X, U).*

$$u_{ij} = \begin{cases} 1 & \textit{if edge } [i,j] \textit{ is present in the } n\textit{-tour}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Consider $1 \leqslant i < j \leqslant 3$ then from Eq. (4.9) we have

$$u_{ij} = 1 - \sum_{k=4}^{n} x_{ijk},$$

$u_{ij} = 0 \Rightarrow \sum_{k=4}^{n} x_{ijk} = 1$, which implies that for some $4 \leqslant k \leqslant n$, $x_{ijk} = 1$, i.e, $[i,j]$ is not in the solution.

Conversely, suppose $[i,j]$ is not in the solution, then we have $\sum_{k=4}^{n} x_{ijk} = 1$ for some $4 \leqslant k \leqslant n$ which implies that $u_{ij} = 0$.

Now consider $1 \leqslant i < j; 4 \leqslant j \leqslant n-1$,

$$u_{ij} = \sum_{1 \leqslant r \leqslant i-1} x_{rij} + \sum_{i+1 \leqslant s \leqslant j-1} x_{isj} - \sum_{j+1 \leqslant k \leqslant n} x_{ijk},$$

$[i,j]$ is not present in the solution if

$$\sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} = 0 \Rightarrow \sum_{j+1\leqslant k\leqslant n} x_{ijk} = 0,$$

$$\text{or} \sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} = 1 \quad \text{and} \quad \sum_{j+1\leqslant k\leqslant n} x_{ijk} = 1.$$

Hence $u_{ij}=0$ if $[i,j]$ is not present in the solution. Conversely, if $u_{ij}=0$ we show that $[i,j]$ is not present in the solution.

$$u_{ij} = 0 \Rightarrow \sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} - \sum_{j+1\leqslant k\leqslant n} x_{ijk} = 0$$

$$\Rightarrow \sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} = \sum_{j+1\leqslant k\leqslant n} x_{ijk}.$$

There are two cases:

(a)

$$\sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} = 0.$$

This implies that edge $[i,j]$ is not created upto the j stage and hence is not available for insertion of $k; j+1\leqslant k\leqslant n$. Hence $[i,j]$ is not in the solution.

(b)

$$\sum_{1\leqslant r\leqslant i-1} x_{rij} + \sum_{i+1\leqslant s\leqslant j-1} x_{isj} = 1 = \sum_{j+1\leqslant k\leqslant n} x_{ijk},$$

which implies that edge $[i,j]$ is created before stage j, but then some $k, j+1\leqslant k\leqslant n$ is inserted between $[i,j]$. Hence, $[i,j]$ is not in the solution.

Hence, $u_{ij}=0$ *iff* $[i,j]$ is not in the solution.

Lemma 4.2. *Corresponding to any feasible solution to Problem* 2, *we have*
(i) $\sum_{1\leqslant i<j\leqslant n} u_{ij} = n$, *and*
(ii) $\forall 1\leqslant i<j\leqslant n, 0\leqslant u_{ij}\leqslant 1$.

Proof. (i) We shall show that this is true for any feasible solution (X,U) to Problem 2. As (X,U) is feasible we have

$$\boldsymbol{E_n}X = e_{n-3}, \tag{4.15}$$

$$\boldsymbol{A}X + \boldsymbol{I}U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix}. \tag{4.16}$$

Now sum the last $(n(n-1))/2$ terms of (4.16). We get

$$-\sum_{k=4}^{n} \sum_{1\leqslant i<j\leqslant k-1} x_{ijk} + \sum_{1\leqslant i<j\leqslant n} u_{ij} = 3. \tag{4.17}$$

But $\sum_{k=4}^{n}\sum_{1\leqslant i<j\leqslant k-1}x_{ijk}=n-3$ as obtained from the sum of the first $(n-3)$ rows of (4.15).

(ii) *Case* (a): $1\leqslant i<j\leqslant 3$.

We have $0\leqslant x_{ijk}\Rightarrow 0\leqslant\sum_{k=4}^{n}x_{ijk}$.

Also $\sum_{k=4}^{n}x_{ijk}\leqslant 1$. Hence we have

$$1\geqslant 1-\sum_{k=4}^{n}x_{ijk}\geqslant 0$$

$$\Rightarrow 0\leqslant u_{ij}\leqslant 1.$$

Case (b): $1\leqslant i<j; 4\leqslant j\leqslant n-1$

$$u_{ij}=\sum_{1\leqslant r\leqslant i-1}x_{rij}+\sum_{i+1\leqslant s\leqslant j-1}x_{isj}-\sum_{j+1\leqslant k\leqslant n}x_{ijk}.$$

Hence

(i) $\sum_{1\leqslant r\leqslant i-1}x_{rij}+\sum_{i+1\leqslant s\leqslant j-1}x_{isj}=1, \sum_{j+1\leqslant k\leqslant n}x_{ijk}=1\Rightarrow u_{ij}=0,$

(ii) $\sum_{1\leqslant r\leqslant i-1}x_{rij}+\sum_{i+1\leqslant s\leqslant j-1}x_{isj}=1, \sum_{j+1\leqslant k\leqslant n}x_{ijk}=0\Rightarrow u_{ij}=1,$

(iii) $\sum_{1\leqslant r\leqslant i-1}x_{rij}+\sum_{i+1\leqslant s\leqslant j-1}x_{isj}=0, \sum_{j+1\leqslant k\leqslant n}x_{ijk}=0\Rightarrow u_{ij}=0.$

Hence $0\leqslant u_{ij}\leqslant 1$.

Observe that $\mathscr{C}'=(\mathscr{C}_{124},\ldots,\mathscr{C}_{12n},\ldots,\mathscr{C}_{(n-2)(n-1)n})$ is such that $\mathscr{C}'=-c'A$.

Consider any solution (X,U) to Problem 2. Then

$$\boldsymbol{A}X+\boldsymbol{I}U=\begin{bmatrix}e_3\\0\end{bmatrix}.$$

Premultiply both sides by c'. Now

$$c'U=c'\begin{bmatrix}e_3\\0\end{bmatrix}-c'\boldsymbol{A}X$$

$$=c'\begin{bmatrix}e_3\\0\end{bmatrix}+\mathscr{C}'X. \tag{4.18}$$

But $c'\begin{bmatrix}e_3\\0\end{bmatrix}=c_{12}+c_{13}+c_{23}$ is a constant given X. Therefore, it is sufficient to minimise $c'U$ in order to minimise $\mathscr{C}'X$.

Now we have Problem 3 which is equivalent to Problem 2.

Problem 3.

$$\text{minimise}\quad c'U\quad\text{such that}\quad\begin{bmatrix}\boldsymbol{E_n}&\boldsymbol{0}\\\boldsymbol{A}&\boldsymbol{I}\end{bmatrix}\begin{bmatrix}X\\U\end{bmatrix}=\begin{bmatrix}e_{n-3}\\e_3\\0\end{bmatrix}, \tag{4.19}$$

$$X,U\geqslant 0.$$

Remark 4.2. Any n-tour corresponds to an integer basic feasible solution. But there are basic feasible solutions which are non-integer as illustrated in the following example.

Example 4.2. Let $x_{124} = x_{134} = x_{135} = x_{245} = \frac{1}{2}$. There is a basic feasible solution to Problem 3 with corresponding $u_{12} = \frac{1}{2}$, $u_{13} = 0$, $u_{23} = 1$, $u_{14} = 1$, $u_{24} = 0$, $u_{34} = \frac{1}{2}$, $u_{15} = u_{25} = u_{35} = u_{45} = \frac{1}{2}$.

Let

$$\zeta(n) = \{X \backslash \boldsymbol{E_n} X = e_{n-3}, X \geqslant 0\}, \tag{4.20}$$

$$\mathscr{U}(n) = \left\{ U \backslash U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - \boldsymbol{A}X \geqslant 0, X \in \zeta(n) \right\}. \tag{4.21}$$

Remark 4.3. Let

$$U^s = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - \boldsymbol{A}X^s \geqslant 0$$

for any integer $X^s \in \zeta(n)$. Then U^s is an extreme point of $\mathscr{U}(n)$.

Proof. Let $U, V \in \mathscr{U}(n) - \{U^s\}$. We shall show that $\lambda U + (1 - \lambda)V$ *for* $\lambda \in (0, 1)$ belongs to $\mathscr{U}(n) - \{U^s\}$.

We have $U \neq U^s \neq V$.

As $U, V \in \mathscr{U}(n) - \{U^s\}$ there exists X, Y such that

$$U = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AX \geqslant 0$$

and

$$V = \begin{bmatrix} e_3 \\ 0 \end{bmatrix} - AY \geqslant 0.$$

Note that $X \neq X^s \neq Y$.

Now $U + (1 - \lambda)V \in \mathscr{U}(n)$ since $\mathscr{U}(n)$ is a convex set. We want to prove that $\lambda U + (1 - \lambda)V \in \mathscr{U}(n) - \{U^s\}$.

Suppose this is not true.Then $\lambda U + (1 - \lambda)V = U^s$. Since X^s is integer U^s is also integer. We know that

$$\sum_{1 \leqslant p < q \leqslant n} U^s_{pq} = \sum_{1 \leqslant p < q \leqslant n} U_{pq} = \sum_{1 \leqslant p < q \leqslant n} V_{pq} = n$$

as these correspond to feasible solutions to Problem 2. Also note that for any feasible solution (X, U) to Problem 2, $0 \leqslant u_{ij} \leqslant 1$. Therefore if any coordinate of U^s is zero the corresponding coordinates of U as well as V have to be zero, as λ, $(1 - \lambda) > 0,$ *and* $U, V \geqslant 0$.

Thus $U = V = U^s$, which leads to a contradiction as $U, V \in \mathscr{U}(n) - \{U^s\}$. Hence the result.

$\mathscr{U}(n)$ is the orthogonal projection of the polytope $\zeta(n)$. It is expected that some of the projected extreme points are no longer extremal in the projection as shown in the following two examples.

Example 4.3. Consider the fractional basic feasible solution given earlier for the 5-city problem in Example (4.2). This solution can be written as a convex combination of solutions given by

(a) $x_{134} = 1, x_{125} = 1, u_{23} = u_{14} = u_{34} = u_{15} = u_{25} = 1$, and

(b) $x_{134} = x_{345} = 1, u_{12} = u_{23} = u_{14} = u_{35} = u_{45} = 1$ with equal weightage given to both the tours.

Here we have an example of a slack variable vector U corresponding to a fractional basic feasible solution to Problem 3 which need not be an extreme point of $\mathscr{U}(n)$. However, a question still remains is whether the set of all extreme points of $\mathscr{U}(n)$ is the set of all $U's$ corresponding to integer feasible solutions? The answer is NO as shown by the following example.

Example 4.4. Consider the Petersen's graph $G = (V, E)$ where

$V = \{1, 2, \dots, 10\}$,

$E = \{[1,2],[1,5],[1,9],[2,3],[2,7],[3,4],[3,10],[4,5],[4,8],[5,6],[6,7],[7,8],$
$[8,9],[9,10],[6,10]\}$

Let

$$c_{ij} = \begin{cases} -1 & \text{if } [i,j] \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the 10-city STSP on the above graph. It is well known that Petersen's graph is non-Hamiltonian i.e. there is no 10-tour available only using the edges of the graph G. Any tour uses 10 edges of the complete graph K_{10}. So, an optimal tour for this problem will have an objective value of at least -9 since it has to use an edge not in E.

However the following fractional solution to the problem has objective function value -10.

$x_{134} = x_{135} = x_{356} = x_{147} = x_{178} = x_{348} = x_{478} = x_{139} = x_{189} = x_{389} = x_{3,6,10} = \frac{1}{3}$,

$x_{234} = x_{245} = x_{256} = x_{267} = x_{3910} = \frac{2}{3}$,

$u_{12} = u_{56} = u_{310} = 1$,

$u_{34} = u_{45} = u_{27} = u_{67} = u_{48} = u_{78} = u_{19} = u_{89} = u_{910} = \frac{2}{3}$,

$u_{23} = u_{15} = u_{610} = \frac{1}{3}$; and other u_{ij}'s are zeroes. u_{ij} values are shown along the edges in Fig. 1.

As u_{ij} add up to 10 and the distance associated with the edges in the Petersen's graph is -1, we have -10 as the objective function value corresponding to this solution. It is not possible to write this solution as a convex combination of U vectors corresponding to tour solutions, which have objective function value at least -9.

Theorem 4.2. *$\mathscr{U}(n) \subseteq SEP(n)$ where $SEP(n)$ is defined as in Section* 2.

Proof. The proof is by induction on n. Consider constraints (4.8)–(4.12) of Problem 2, other than the non-negativity restrictions (4.11). We introduce the following notation to facilitate the induction proof.

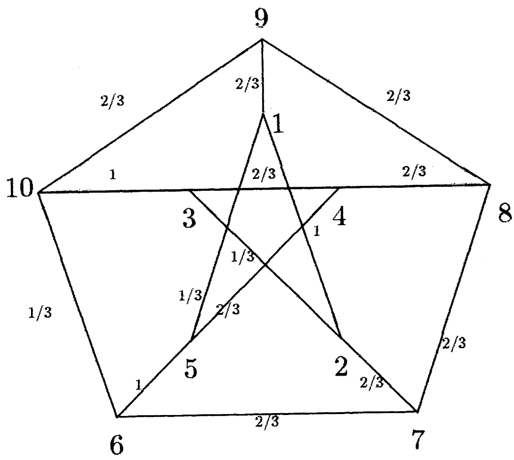

Fig. 1. Petersen graph with values given in Example 4.4.

Let u_{ij}^n be the slack variables associated with the constraint corresponding to the pair$[i, j]$, when we have n cities in all. Recall that $\mathscr{U}(n)$ is the set of all U, such that there exists X, such that (X, U) is feasible for Problem 2. We have introduced a superscript for U now. Let U^n be the vector of slack variables (u_{ij}^n).

We have,

$$\sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} = 1, \quad 4 \leqslant k \leqslant n, \tag{4.22}$$

$$\sum_{k=4}^{n} x_{ijk} + u_{ij}^n = 1, \quad 1 \leqslant i < j \leqslant 3, \tag{4.23}$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n} x_{ijk} + u_{ij}^n = 0, \quad 4 \leqslant j \leqslant n-1,\ 1 \leqslant i < j, \tag{4.24}$$

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-1} x_{isn} + u_{i,n}^n = 0, \quad i = 1, \ldots, n-1. \tag{4.25}$$

Now consider the problem with the number of cities equal to $n-1$, with the first $n-1$ cities. We have the corresponding equality constraints, after introducing u_{ij}^{n-1}, the slack variables,

$$\sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} = 1, \quad 4 \leqslant k \leqslant n-1, \tag{4.26}$$

$$\sum_{k=4}^{n-1} x_{ijk} + u_{ij}^{n-1} = 1, \quad 1 \leqslant i < j \leqslant 3, \tag{4.27}$$

$$-\sum_{r=1}^{i-1} x_{rij} - \sum_{s=i+1}^{j-1} x_{isj} + \sum_{k=j+1}^{n-1} x_{ijk} + u_{ij}^{n-1} = 0, \quad 4 \leqslant j \leqslant n-2;\ 1 \leqslant i < j, \tag{4.28}$$

$$-\sum_{r=1}^{i-1} x_{rin} - \sum_{s=i+1}^{n-2} x_{isn} + u_{i,n-1}^{n-1} = 0, \quad i = 1,\dots,n-2. \tag{4.29}$$

Comparing these two sets of constraints, we notice that, given a non-negative solution (X, U^n) for the *n-city* problem, we have, $(X/(n-1), U^{n-1})$ given below is a non-negative solution to the problem with first $(n-1)$ cities:

$$X/(n-1) = (x_{123}, \dots, x_{n-3,n-2,n-1}), \tag{4.30}$$

$$u_{ij}^{n-1} = u_{ij}^{n} + x_{ijn} \forall 1 \leqslant i < j \leqslant n-2. \tag{4.31}$$

Basis for induction. We first prove that the result is true for $n=4$. i.e. $\mathscr{U}(4) \subseteq SEP(4)$. We have u_{ij}^4 as the slack variables. From Eq. (4.22)–(4.25) we have the following:

$$u_{ij}^4 = 1 - x_{ij4};\ 1 \leqslant i < j \leqslant 3, \tag{4.32}$$

$$u_{i4}^4 = \sum_{r=1}^{i-1} x_{ri4} + \sum_{s=i+1}^{3} x_{is4};\ 1 \leqslant i \leqslant 3. \tag{4.33}$$

Notice that all u_{ij}^4 are non-negative. Now we show that the degree constraints (2.2) are satisfied for all i.

$i = 1$:

$$u_{12}^4 + u_{13}^4 + u_{14}^4 = 1 - x_{124} + 1 - x_{134} + x_{124} + x_{134} = 2. \tag{4.34}$$

Similarly checked for $i = 2$ and 3.

$i = 4$:

$$u_{14}^4 + u_{24}^4 + u_{34}^4 = x_{124} + x_{134} + x_{124} + x_{234} + x_{134} + x_{234} = 2. \tag{4.35}$$

Since the subtour elimination constraints in cut form (2.3) for $(V \backslash S)$ are implied by the subtour elimination constraints for S we verify that the subtour elimination constraints for $|S| = 2, S \subseteq V_4$ are satisfied.

Let i_1, i_2, i_3 be a permutation of $(1,2,3)$.

$$u^4(\delta(S)) = 2 + 2x_{i_1 i_2 4} \quad \text{for } S = \{i_1, i_2\} \text{ Or } S = \{i_3, 4\}. \tag{4.36}$$

Thus we have, $u^4(\delta(S)) \geqslant 2$ as $x_{i_1 i_2 4} \geqslant 0$. Hence for $n = 4$ we have the result.

Let us assume $\mathscr{U}(n-1) \subseteq SEP(n-1)$. We shall show that $\mathscr{U}(n) \subseteq SEP(n)$. Since we are going to deal with value of cut corresponding to subsets of V_n, hereon we assume symmetry of the notation of subscripts denoting the edges, i.e, $u_{ij} = u_{ji}$. We show that constraints (2.3) hold for the required nonempty proper subsets S of V_n. i.e. $|S|$ is between 2 and $\lfloor V_n \rfloor + 1$. Let

$$\delta^n = \sum_{r \in S, s \in \bar{S}} u_{rs}^n = u^n(\delta(S)), \tag{4.37}$$

be the value of the cut corresponding to a subset S, given (X, U^n) feasible for the $n-city$ problem, with $U^n \in \mathscr{U}(n)$. We need to show that

$$\delta^n \geqslant 2. \tag{4.38}$$

Without loss of generality, let $n \in \bar{S}$.

Define

$$P = \{[i,j] \backslash x_{ijn} > 0\}. \tag{4.39}$$

Let $S = \{i_1, i_2, \ldots, i_m\}$ and $\bar{S} = \{j_1, j_2, \ldots, j_l, n\}$.

Now consider U^{n-1} derived from U^n and X. We have by the feasibility of U^{n-1}, $U^{n-1} \in \mathscr{U}(n-1)$. And by induction hypothesis $U^{n-1} \in SEP(n-1)$. Therefore we have,

$$\delta^{n-1} = \sum_{r=1}^{m} \sum_{s=1}^{l} u_{i_r j_s}^{n-1} \geqslant 2. \tag{4.40}$$

We need to show that

$$\delta^n = \sum_{r=1}^{m} \sum_{s=1}^{l} u_{i_r j_s}^{n} + \sum_{r=1}^{m} u_{i_r n}^{n} \geqslant 2. \tag{4.41}$$

Take any point $i_r \in S$. We have

$$\sum_{s=1}^{l} u_{i_r j_s}^{n} = \sum_{[i_r, j_s] \in P} u_{i_r j_s}^{n} + \sum_{[i_r, j_s] \notin P} u_{i_r j_s}^{n}. \tag{4.42}$$

If $[i_r, j_s] \in P$ then $x_{i_r j_s n} > 0$ and

$$u_{i_r j_s}^{n} = u_{i_r j_s}^{n-1} - x_{i_r j_s n}, \tag{4.43}$$

$$\text{otherwise } u_{i_r j_s}^{n} = u_{i_r j_s}^{n-1}. \tag{4.44}$$

Hence

$$\sum_{s=1}^{l} u_{i_r j_s}^{n} = \sum_{[i_r, j_s] \in P} u_{i_r j_s}^{n-1} - \sum_{[i_r, j_s] \in P} x_{i_r j_s n} + \sum_{[i_r, j_s] \notin P} u_{i_r j_s}^{n-1}, \tag{4.45}$$

$$u_{i_r n}^{n} = \sum_{[i_r, i_q] \in P} x_{i_r i_q n} + \sum_{[i_r, j_s] \in P} x_{i_r j_s n}, \tag{4.46}$$

$$\begin{aligned} \sum_{s=1}^{l} u_{i_r j_s}^{n} + u_{i_r n}^{n} = & \sum_{[i_r, j_s] \in P} u_{i_r j_s}^{n-1} - \sum_{[i_r, j_s] \in P} x_{i_r j_s n} \\ & + \sum_{[i_r, j_s] \notin P} u_{i_r j_s}^{n-1} + \sum_{[i_r, i_q] \in P} x_{i_r i_q n} + \sum_{[i_r, j_s] \in P} x_{i_r j_s n}. \end{aligned} \tag{4.47}$$

Therefore,

$$\delta^n = \sum_{r=1}^{m}\left[\sum_{s=1}^{l} u_{i_r j_s}^{n-1} + \sum_{[i_r,i_q]\in P} x_{i_r i_q n}\right] \tag{4.48}$$

$$= \delta^{n-1} + \sum_{r=1}^{m}\sum_{[i_r,i_q]\in P} x_{i_r i_q n} \geqslant 2. \tag{4.49}$$

Hence $\delta^n \geqslant 2\ \forall n$.

We can check that the degree constraints are satisfied, as follows:

If S is a singleton set, say $S=\{i\}, i\neq n$, then $u^n(\delta(S)) = u^n(\delta(i))$ is still greater than or equal to 2 , as the preceding arguments go through for $m=1$, the cardinality of S. However, notice that for no i the strict inequality can hold, as it will contradict the fact $\sum_{1\leqslant i<j\leqslant n} u_{ij}^n = n$.

Hence the theorem.

5. Conclusions

We now have a formulation which uses only polynomial number of constraints and defines $SEP(n)$. Padberg and Sung [9] mention that all the asymmetric formulations which when symmetrised result in constraint sets of exponential sizes. Therefore, now we have a symmetric formulation which has a constraint set of polynomial size. The advantage of this formulation is that we can start with this formulation as well and add all other facet defining inequalities to this constraint set. Implications of this formulation in optimising over $SEP(n)$ and other issues are studied elsewhere by the authors.

Using the techniques discussed in Padberg and Sung we have shown in Arthanari and Usha [2] that $\mathscr{U}(n) \equiv SEP(n)$. We obtain the linear description of $\mathscr{U}(n)$ as well in that paper.

Acknowledgements

The authors are grateful to the anonymous referee for the useful comments and suggestions towards improving the clarity of presentation of this paper.

References

[1] T.S. Arthanari, On the traveling salesman problem, Presented at the XI Symposium on Mathematical Programming held at Bonn, West Germany, 1982.

[2] T.S. Arthanari, M. Usha, Equivalence of $\mathscr{U}(n)$ and $SEP(n)$ and its implications, Preprint, communicated for publication, 1997.

[3] R.E. Bellman, Dynamic programming treatment of the travelling salesman problem, J. Assoc. Comput. Mach. 9 (1962) 61–63.

[4] G. Dantzig, D. Fulkerson, S. Johnson, Solution of a large scale traveling salesman problem, Oper. Res. 2 (1954) 393–410.
[5] L. Gouveia, S. Voß, A classification of formulations for the (time-dependent) traveling salesman problem, Eur. J. Oper. Res. 83 (1995) 69–82.
[6] M. Held, R.M. Karp, A dynamic programming approach to sequencing problems, SIAM Appl. Math. 10 (1962) 196–210.
[7] M. Junger et al., The traveling salesman problem, Handbooks in OR & MS, Vol. 7, Elseiver Science, Amsterdam, 1995.
[8] E. Lawler et al. (eds)., The traveling salesman problem, Wiley, New York, 1985.
[9] M. Padberg, T.Y. Sung, An analytical comparison of different formulations of the traveling salesman problem, Math. Prog. 52 (1991) 315–357.